

Gleichung (1):

$$\begin{aligned}
 H|0\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot 1 + \frac{1}{\sqrt{2}} \cdot 0 \\ \frac{1}{\sqrt{2}} \cdot 1 + \left(-\frac{1}{\sqrt{2}}\right) \cdot 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \\
 &= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)
 \end{aligned}$$

Gleichung (2):

$$\begin{aligned}
 H\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) &= H\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle
 \end{aligned}$$

Gleichung (3):

$${}_c\text{NOT}|01\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 \\ 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 \\ 0 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 \\ 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle$$

Gleichung (4):

$${}_c\text{NOT}|10\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 \\ 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 \\ 0 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 \\ 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle$$

Gleichung (5):

$$\begin{aligned}
 H \otimes I = HI &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & -\frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot 1 & \frac{1}{\sqrt{2}} \cdot 0 & \frac{1}{\sqrt{2}} \cdot 1 & \frac{1}{\sqrt{2}} \cdot 0 \\ \frac{1}{\sqrt{2}} \cdot 0 & \frac{1}{\sqrt{2}} \cdot 1 & \frac{1}{\sqrt{2}} \cdot 0 & \frac{1}{\sqrt{2}} \cdot 1 \\ \frac{1}{\sqrt{2}} \cdot 1 & \frac{1}{\sqrt{2}} \cdot 0 & -\frac{1}{\sqrt{2}} \cdot 1 & -\frac{1}{\sqrt{2}} \cdot 0 \\ \frac{1}{\sqrt{2}} \cdot 0 & \frac{1}{\sqrt{2}} \cdot 1 & -\frac{1}{\sqrt{2}} \cdot 0 & -\frac{1}{\sqrt{2}} \cdot 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}
 \end{aligned}$$

Zusatzgleichung (5-2) für Interessierte: Falls die beiden Operationen vertauscht wären, also die Identitätsmatrix auf x_0 und das Hadamard Gatter auf x_1 angewendet würde, dann sähe die kombinierte Operation so aus:

$$\begin{aligned}
 I \otimes H = IH &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} & 0 \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\ 0 \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} & 1 \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \end{bmatrix} \\
 &= \begin{bmatrix} 1 \cdot \frac{1}{\sqrt{2}} & 1 \cdot \frac{1}{\sqrt{2}} & 0 \cdot \frac{1}{\sqrt{2}} & 0 \cdot \frac{1}{\sqrt{2}} \\ 1 \cdot \frac{1}{\sqrt{2}} & 1 \cdot \left(-\frac{1}{\sqrt{2}}\right) & 0 \cdot \frac{1}{\sqrt{2}} & 0 \cdot \left(-\frac{1}{\sqrt{2}}\right) \\ 0 \cdot \frac{1}{\sqrt{2}} & 0 \cdot \frac{1}{\sqrt{2}} & 1 \cdot \frac{1}{\sqrt{2}} & 1 \cdot \frac{1}{\sqrt{2}} \\ 0 \cdot \frac{1}{\sqrt{2}} & 0 \cdot \left(-\frac{1}{\sqrt{2}}\right) & 1 \cdot \frac{1}{\sqrt{2}} & 1 \cdot \left(-\frac{1}{\sqrt{2}}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}
 \end{aligned}$$

Gleichung (6):

$$\begin{aligned}
 HI|00\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot 1 + 0 \cdot 0 + \frac{1}{\sqrt{2}} \cdot 0 + 0 \cdot 0 \\ 0 \cdot 1 + \frac{1}{\sqrt{2}} \cdot 0 + 0 \cdot 0 + \frac{1}{\sqrt{2}} \cdot 0 \\ \frac{1}{\sqrt{2}} \cdot 1 + 0 \cdot 0 + -\frac{1}{\sqrt{2}} \cdot 0 + 0 \cdot 0 \\ 0 \cdot 1 + \frac{1}{\sqrt{2}} \cdot 0 + 0 \cdot 0 + -\frac{1}{\sqrt{2}} \cdot 0 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)
 \end{aligned}$$

Zusatzgleichung (6-2) für Interessierte:

$$\begin{aligned}
 IH|00\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot 1 + \frac{1}{\sqrt{2}} \cdot 0 + 0 \cdot 0 + 0 \cdot 0 \\ \frac{1}{\sqrt{2}} \cdot 1 + -\frac{1}{\sqrt{2}} \cdot 0 + 0 \cdot 0 + 0 \cdot 0 \\ 0 \cdot 1 + 0 \cdot 0 + \frac{1}{\sqrt{2}} \cdot 0 + \frac{1}{\sqrt{2}} \cdot 0 \\ 0 \cdot 1 + 0 \cdot 0 + \frac{1}{\sqrt{2}} \cdot 0 + -\frac{1}{\sqrt{2}} \cdot 0 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)
 \end{aligned}$$

Gleichung (7):

$$\begin{aligned}
 cNOT \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot \frac{1}{\sqrt{2}} + 0 \cdot 0 + 0 \cdot \frac{1}{\sqrt{2}} + 0 \cdot 0 \\ 0 \cdot \frac{1}{\sqrt{2}} + 1 \cdot 0 + 0 \cdot \frac{1}{\sqrt{2}} + 0 \cdot 0 \\ 0 \cdot \frac{1}{\sqrt{2}} + 0 \cdot 0 + 0 \cdot \frac{1}{\sqrt{2}} + 1 \cdot 0 \\ 0 \cdot \frac{1}{\sqrt{2}} + 0 \cdot 0 + 1 \cdot \frac{1}{\sqrt{2}} + 0 \cdot 0 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)
 \end{aligned}$$